Structure of JT boundary correlation functions

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Based on arXiv:2006.07072 with G.J. Turiaci
arXiv:2007.00998
arXiv:2106.09353 with Y. Fan
WIP

Outline

Introduction

JT boundary two-point function
- Schwarzian description and perturbative treatment

Gauge theory perspective: $SL(2, \mathbb{R})$ BF model
- Exact treatment
- Special bilocal correlators

String theory perspective: Liouville gravity and minimal string
- Quantum group deformation interpretation
- Higher topology

Conclusion

Introduction

Many developments in lower-dimensional (Jackiw-Teitelboim (JT)) gravity:

- Much focus has been on spectral properties (partition function, spectral form factor) exhibiting chaotic features
  Higher genus and multi-boundary amplitudes: important to understand very-late time behavior, replica wormholes ...

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  Emphasis on on their structure and relation to solvable models
  For interpretation in terms of gravitational physics, see e.g. shockwave scattering TM-Turiaci-Verlinde '18, bulk reconstruction Blommaert-TM-Verschelde '19-'20, TM '19, geodesic lengths and complexity Yang '18...

JT Quantum Gravity as Schwarzian QM (1)

Jackiw-Teitelboim (JT) 2d dilaton gravity Teitelboim '83, Jackiw '85

$S = \frac{1}{16\pi G}\int d^2x\sqrt{-g}\Phi(R-\Lambda) + \frac{1}{8\pi G}\oint d\tau\sqrt{-\gamma}\Phi_{bdy}K$

$\Phi =$ dilaton field

$\Lambda = -2 \rightarrow$ aAdS space $\rightarrow$ holography

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No local dof in 1+1d gravity $\rightarrow$ bulk is topological

With suitable boundary conditions (asymptotic Poincaré, constant boundary value of dilaton Φ), description in terms of dynamical holographic boundary curve with Schwarzian action:

$\Rightarrow S = -C \int d\tau \{F, \tau\}, \quad C \sim \frac{1}{G}, \ \{F, \tau\} = \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2$

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$F(\tau)$ dynamical time reparametrization in terms of proper time τ

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Derivation to be compared to 3d Chern-Simons / 2d WZW CFT duality where would-be large gauge dofs of $A_\mu = g^{-1}\partial_\mu g$ are identified with the physical dof g in the WZW model

JT Quantum Gravity as Schwarzian QM (2)

Transfer to thermal theory and obtain boundary correlation functions of JT gravity / Schwarzian QM:

$$\boxed{\left\langle \mathcal{O}_{h_1}\mathcal{O}_{h_2}\ldots\right\rangle_\beta = \frac{1}{Z}\int_{\mathcal{M}}[\mathcal{D}f]\mathcal{O}_{h_1}\mathcal{O}_{h_2}\ldots\; e^{C\int_0^\beta d\tau\,\{F,\tau\}}}$$

with $F \equiv \tan\left(\frac{\pi f(\tau)}{\beta}\right)$, $\qquad \left\{F,\tau\right\} = \left\{f,\tau\right\} + \frac{2\pi^2}{\beta^2}f'^2$

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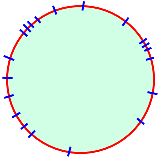
$f(\tau)$ is dynamical reparametrization of S^1:
$f(\tau+\beta) = f(\tau)+\beta, \quad \dot{f} \geq 0$



Red: holographic boundary
Blue: clock ticking pattern for a specific off-shell choice of $f(\tau)$

Boundary bilocal operator

Natural class of bilocal operators:

$$\boxed{\mathcal{O}_h(\tau_1, \tau_2) \equiv \left(\frac{F'(\tau_1)F'(\tau_2)}{(F(\tau_1) - F(\tau_2))^2} \right)^h}$$

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View as reparametrized boundary CFT two-point function coupled to the dynamical time variable $F(\tau)$

Also, interpret as result of worldline path integral of scalar particle of mass $m^2 = h(h-1)$ emitted and absorbed at the boundary at times τ_1 and τ_2

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Interpretation in terms of boundary graviton $\epsilon(\tau)$ and their interactions Kitaev '15, Maldacena-Stanford '16, Maldacena-Stanford-Yang '16, . . .

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$\langle\epsilon(\tau)\epsilon(0)\rangle = \frac{1}{2\pi C}\left[-\frac{1}{2}(u-\pi)^2 + (\tau-\pi)\sin u + 1 + \frac{\pi^2}{6} + \frac{5}{2}\cos u\right]$

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Operator: $\left(\frac{F'(\tau_1)F'(\tau_2)}{(F(\tau_1)-F(\tau_2))^2}\right)^h = \frac{(1+\epsilon_1')^h(1+\epsilon_2')^h}{(\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(\tau_1-\tau_2+\epsilon_1-\epsilon_2))^{2h}}$

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Higher order corrections get complicated very quickly since:

- More vertices from Schwarzian action
- Non-trivial path-integral measure

Gauge theory formulation of JT gravity: the BF model

Gauge theory perspective gives exact approach: 1^{st} order formulation of JT gravity (without boundaries) is given in terms of $SL(2,\mathbb{R})$ BF theory Fukuyama-Kamimura '85, Isler-Trugenberger '89, Chamseddine-Wyler '89

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Path integrate out $B \Rightarrow A = gdg^{-1}$
$\longrightarrow$ Particle on group manifold G:
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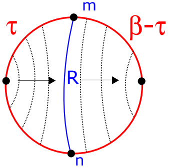
Structure of theory:

- Hilbert space is determined by Peter-Weyl theorem:
  $\mathcal{H} = \{|R,a,b\rangle\,,\ R = \text{ unitary irrep of } G,\ a,b = 1..\mathrm{dimR}\}$
  Hamiltonian eigenstates: $\hat{H}\,|R,a,b\rangle = \mathcal{C}_R\,|R,a,b\rangle$
- Coordinate basis $\{|g\rangle\,, g \in G\}$ with overlap
  $\langle g|R,ab\rangle = \sqrt{\mathrm{dim\ R}}\,R_{ab}(g)$

BF Wilson line

Consider disk amplitude with a boundary-anchored Wilson line:



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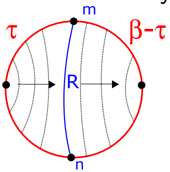

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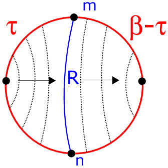

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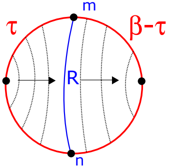

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Using $\langle g|R, ab\rangle = \sqrt{\text{dim R}}\, R_{ab}(g)$ and the group integral:

$$\int dg\, R_{1,m_1 n_1}(g) R_{2,m_2 n_2}(g) R_{3,m_3 n_3}(g) = \begin{pmatrix} R_1 & R_2 & R_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} R_1 & R_2 & R_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

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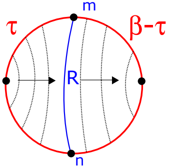

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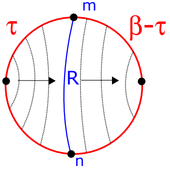

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Remark: Structure of amplitudes well-known from 2d YM

Gravity from SL(2,$\mathbb{R}$): Path-integral relation

Gravity: SL$(2,\mathbb{R})$ group element g with (gravitational) constraint at the holographic boundary:

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- Operator insertions:
  Plugging this into the boundary-anchored Wilson line in lowest weight state of discrete infinite-dimensional irrep $j=-h$ of $\mathrm{SL}(2,\mathbb{R})$:
  $\mathcal{W}^R_{00} = \ldots = \left(\frac{F'(\tau_1)F'(\tau_2)}{(F(\tau_1)-F(\tau_2))^2}\right)^h$

Gravity from SL(2,$\mathbb{R}$): Amplitudes

States:
Hilbert space spanned by constrained (mixed parabolic) matrix elements, or Whittaker functions: $R^k_{00}(\phi) = e^{\phi} K_{2ik}\left(e^{\phi}\right)$

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$$\begin{pmatrix} k_1 \; h \; k_2 \\ 0 \;\; 0 \;\; 0 \end{pmatrix}^2 = \int_{-\infty}^{+\infty} d\phi \, K_{2ik_1}(e^{\phi}) \, e^{2h\phi} \, K_{2ik_2}(e^{\phi}) = \frac{\Gamma(h \pm ik_1 \pm ik_2)}{\Gamma(2h)}$$

JT gravity boundary two-point function

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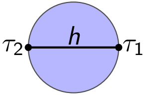
$\langle \mathcal{O}_h(\tau_1, \tau_2) \rangle_\beta =$ 

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$$\int_0^{+\infty} dk_1 (k_1 \sinh 2\pi k_1) dk_2 (k_2 \sinh 2\pi k_2)\, e^{-\tau k_1^2 - (\beta - \tau) k_2^2} \, \frac{\Gamma\left(h \pm i k_1 \pm i k_2\right)}{\Gamma(2h)}$$

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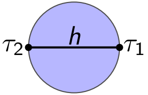

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Other approaches:

- 1d Liouville $f' = e^{\phi}$ Bagrets-Altland-Kamenev '16, '17
- 2d Liouville CFT between ZZ-branes with Liouville primary operator insertions TM-Turiaci-Verlinde '17, TM '18
- Particle in infinite B-field in AdS_2 Yang '18, Kitaev-Suh '18
- $\rightarrow$ Minimal string / Liouville gravity as q-deformation of the BF perspective TM-Turiaci '19, '20, TM '20

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Reinstate Schwarzian coupling C in expression:

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  Exception: $2h \in -\mathbb{N}$
  Correlator is zero unless $k_1 \pm k_2 \in i\mathbb{N}$ $\rightarrow$ along codimension-1 slice

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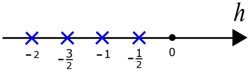
$\Rightarrow$ Convergent $1/C$ expansion 

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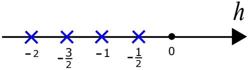
  $\Rightarrow$ Convergent $1/C$ expansion 

- Binomial factors match with binomial expansion of bilocal operator $\left(\frac{(F(\tau_1) - F(\tau_2))^2}{F'(\tau_1) F'(\tau_2)} \right)^j$

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String theory perspective by embedding JT in Liouville gravity

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- For most of talk: S_M = arbitrary CFT with $c_M < 1$
  E.g.: (q, p) minimal model: $b^2 = q/p$ minimal string
- $S_{\rm gh}$ is usual bc-ghost theory with $c_{\rm gh} = -26$

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Interested in holography $\rightarrow$ manifold with boundary, of fixed length

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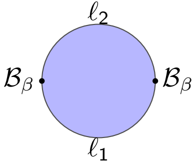
In path integral $\int_{i\mathbb{R}} d\mu_B e^{\mu_B \ell} \times e^{-S_L + S_\partial}$ yields $\delta\left(\ell - \oint e^{b\phi}\right)$, a delta-function on boundary length

Generalization: piecewise constant μ_B allows boundary with fixed length segments $\ell_1, \ldots \ell_n$

Boundary two-point function

Consider two boundary tachyon vertex operators, obtained by dressing a matter primary Φ_M with the Liouville and bc ghost

$\langle \mathcal{B}_\beta \mathcal{B}_\beta \rangle_{\ell_1,\ell_2} =$  $\qquad \mathcal{B}_\beta = c\, \Phi_M e^{\beta\phi}$

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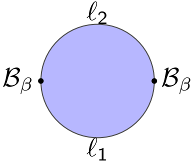

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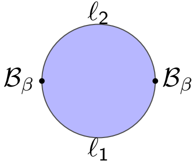

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JT limit ($b \to 0$, $\beta_M = bh$, $S_b(bx) \sim \Gamma(x)$):

$$\int_0^{+\infty} dk_1\, dk_2\, (k_1 \sinh 2\pi k_1)\,(k_2 \sinh 2\pi k_2)\, e^{-k_1^2 \ell_{\mathrm{JT}1}}\, e^{-k_2^2 \ell_{\mathrm{JT}2}}\, \frac{\Gamma(h \pm ik_1 \pm ik_2)}{\Gamma(2h)}$$

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$$e^{\pi i 2 s x} \int_{-\infty}^{+\infty} \frac{d\zeta}{(2\pi b)^{-2i\zeta/b - 2is/b}} S_b(-i\zeta) S_b(-i2s - i\zeta) e^{-\pi i \epsilon(\zeta^2 + 2s\zeta)} e^{2\pi i \zeta x}$$

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Leads to correct 3j-symbol with two such insertions and one discrete irrep insertion:

$$\int_{-\infty}^{+\infty} dx\, R^{\epsilon}_{s_1,00}(x) e^{2\beta_M \pi x} R^{\epsilon *}_{s_2,00}(x) \sim \frac{S_b(\beta_M \pm i s_1 \pm i s_2)}{S_b(2\beta_M)}$$

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Minimal string: has only primaries from degenerate matter Virasoro reps: $\beta_{M(n,m)} = -\frac{(n-1)b}{2} + \frac{(m-1)}{2b}$ where $n = 1, \ldots, p-1$ and $m = 1, \ldots q-1 \rightarrow$ finite discrete set

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If we further specify to $(2,p)$ minimal string, we have $\beta_M = -bj$, $j = 0, \frac{1}{2}, 1 \ldots$ and Liouville boundary two-point function is simplified into:

$$\frac{1}{S_b(b+bj \pm is_1 \pm is_2)} = \frac{\cosh \frac{2\pi}{b} s_1 + (-)^{2j+1} \cosh \frac{2\pi}{b} s_2}{4^j \prod_{n=-j}^{j} (\cosh 2\pi b s_1 - \cosh 2\pi b(s_2 + inb))}$$

Minimal string boundary two-point function (2)

Transform to fixed-length basis leads to somewhat simpler expression:

$$\int_0^{+\infty} ds \rho_0(s)\, e^{-\ell_1 \cosh 2\pi b s} \sum_{n=-j}^{+j} \frac{(2j)!\, e^{-\ell_2 \cosh 2\pi b(s+inb)}}{\prod_{\substack{m=-j \\ m\neq n}}^{j} (\cosh 2\pi b(s+inb) - \cosh 2\pi b(s+imb))}$$

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Due to this origin in the minimal string, these special operators form an integrable subclass of operators in JT gravity

Other JT operator insertions ($h \notin -\mathbb{N}/2$) are outside this class

Boundary two-point function: higher topology

How do higher genus corrections work for the boundary two-point function?

For partition function, multiboundary amplitudes and spectral form factor, this was studied in Saad-Shenker-Stanford '19

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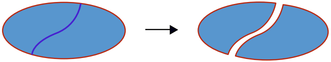
How do we reproduce this from higher topology contributions?

Boundary two-point function: higher topology

Geometrically: Blommaert-TM-Verschelde '19, Saad '19

Rule: **1.** cut surface along the Wilson line:



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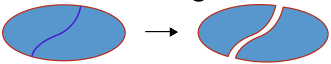

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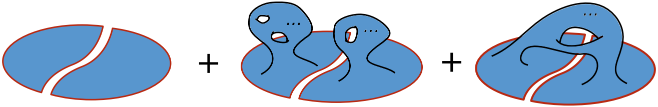

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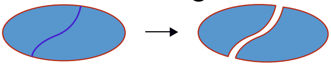

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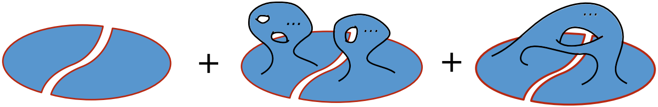

First two diagrams: disconnected pieces: $\rho(E_1)\rho(E_2)$

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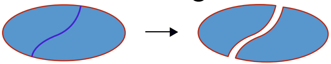

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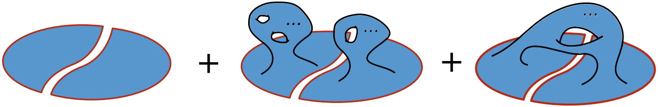

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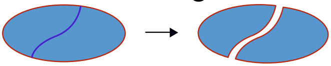

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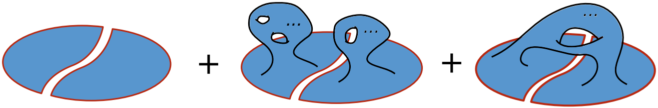

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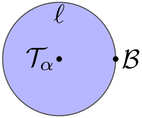
$\rightarrow$ We will use the embedding in Liouville gravity / minimal string where calculations are in principle well-understood

Puzzle: Self-intersecting lines

Start with Liouville gravity bulk-boundary two-point function in fixed length basis TM-Turiaci '20

$\mathcal{T}_\alpha = c\bar{c}\,\mathcal{O}_M e^{2\alpha\phi}, \qquad \alpha = Q/2 + iP,$



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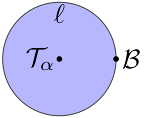

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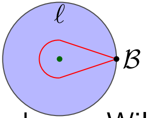

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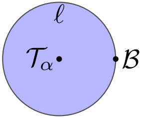

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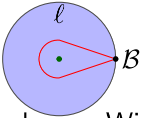

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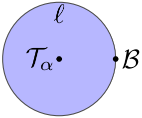

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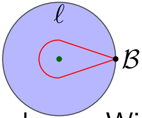

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We have found this conclusion by taking the JT limit from string theory where no Wilson lines are drawn in the first place

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JT gravity is part of a web of exactly solvable models (BF model, minimal string, 3d pure gravity, 3d Chern-Simons ...), for which JT and the Schwarzian model were not studied in the early literature

Hope: leverage this knowledge to learn about quantum gravity!

Thank you!