Generalized Affine $\mathfrak{sl}(2)$ Gaudin Model

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based on joint work with Gleb Kotousov
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Local IM in 2D integrable QFT

- 1+1 QFT with infinite number of continuity equations

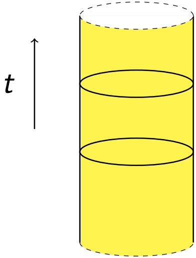
$$\partial_\mu T^{\mu\nu_1\ldots\nu_s} = 0$$

Light cone co-ordinates $x_\pm = t \pm x$

$$\partial_- T_{s+1} = \partial_+ \Theta_{s-1}\,, \qquad \partial_+ T_{-s+1} = \partial_- \Theta_{-s+1} \qquad\qquad (s \geq 0)$$

- For finite volume QFT with $x \sim x + R$

$$\mathbb{I}_s = \int_0^R \mathrm{d}x \big(T_{s+1} - \Theta_{s-1} \big)$$



Integrability assumes

$$\big[\mathbb{I}_s ,\, \mathbb{I}_{s'} \big] = 0$$

Local IM in 2D integrable CFT [A. Zamolodchikov '89]

- Suppose theory is a deformation of the CFT governing its UV fixed point.

  As $r \equiv R/R_{\rm c} \to 0$ the conserved currents become chiral

$$\partial_- T^{(\rm CFT)}_{s+1} = 0\,, \qquad \partial_+ T^{(\rm CFT)}_{-s-1} = 0 \qquad (s \geq 0)$$

- $$\lim_{R\to 0} \underbrace{\big[R^s\, \mathbb{I}_s\big](r)}_{\text{dimensionless}} = \mathbb{I}^{(\rm CFT)}_s\,, \qquad \mathbb{I}^{(\rm CFT)}_s = \int_0^{2\pi} \frac{{\rm d}u}{2\pi}\, T^{(\rm CFT)}_{s+1}$$

$$\big(u = \tfrac{2\pi}{R}\ x\big)$$

- $\mathbb{I}^{(\rm CFT)}_s$ form an infinite commuting family that act in the chiral CFT spaces, which can be classified w.r.t. the irreps of the (extended) conformal algebra (Virasoro, W-algebra, Kac-Moody,...)

Integrable hierarchies

The sets $\{\mathbb{I}_s\}$ are associated with integrable hierarchies

- KdV/sine Gordon

$$\mathbb{I}_{2n-1}\,, \quad n = 1, 2, 3$$

- Non-linear Shrödinger (AKNS)/complex sine Gordon

$$\mathbb{I}_s\,, \quad s = 1, 2, 3$$

- We'll discuss a certain set of mutually commuting local IM:

$$\mathbb{I}^{(a)}_{2n-1} = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\; T^{(a)}_{2n} \qquad a = 1, \ldots, r\ ; \quad n = 1, 2, 3, \ldots\ ,$$

Gaudin model [Gaudin'76]

- Quantum spins $\vec{S}^{(a)} = \big(S_1^{(a)}, S_2^{(a)}, S_3^{(a)}\big)$ $(a = 1, \ldots, r)$:

$$\big[S_A^{(a)}, S_B^{(b)}\big] = \mathrm{i}\,\delta_{ab}\ \varepsilon_{ABC}\ S_C^{(a)}\ , \qquad\qquad \big(\vec{S}^{(a)}\big)^2 = \mathrm{j}_a(\mathrm{j}_a + 1)$$

- Hamiltonians

$$\mathbf{H}^{(a)} = 2 \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{\vec{S}^{(a)} \cdot \vec{S}^{(b)}}{z_a - z_b} \quad :$$

$$\sum_{a=1}^{r} \mathbf{H}^{(a)} = 0\ , \qquad [\mathbf{H}^{(a)}, \mathbf{H}^{(b)}] = 0$$

Bethe Anzatc Equations [Gaudin'76]

- $$\mathbf{H}^{(a)} = 2 \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{\vec{S}^{(a)} \cdot \vec{S}^{(b)}}{z_a - z_b}\ , \qquad \qquad \big(\vec{S}^{(a)}\big)^2 = \mathtt{j}_a(\mathtt{j}_a + 1)$$

- $$\sum_{a=1}^{r} \frac{\mathtt{j}_a}{z_a - x_m^{(+)}} - \sum_{\substack{n=1 \\ n\neq m}}^{\mathtt{M}_+} \frac{1}{x_n^{(+)} - x_m^{(+)}} = 0 \qquad \qquad (m = 1, 2, \ldots, \mathtt{M}_+)$$

  $\mathtt{M}_+ = 0, 1, \ldots, 2\sum_{a=1}^{r} \mathtt{j}_a$.

- $$E_a = \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{2\mathtt{j}_a \mathtt{j}_b}{z_a - z_b} - \sum_{m=1}^{\mathtt{M}_+} \frac{2\mathtt{j}_a}{z_a - x_m^{(+)}}$$

- The Gaudin model admits a generalization to any simple Lie algebra $\mathfrak{g}$ [Gaudin'83;Jurčo'89; Feigin,Frenkel,Reshetikhin'94,...]

- The development of the mathematical apparatus of $2D$ CFT led to the idea that there should be a meaningful generalization to the case when the finite-dimensional Lie algebra is replaced by an Kac-Moody algebra $\widehat{\mathfrak{g}}$.

- The diagonalization problem would be formulated for an infinite set $\{\mathbb{I}_s\}$ of local IM which depend on the arbitrary parameters $\{z_a\}_{a=1}^r$:

$$\mathbb{I}_s = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\ T_{s+1}(u)\ \ : \qquad [\mathbb{I}_s, \mathbb{I}_{s'}] = 0$$

T_{s+1} is a chiral local field of Lorentz spin $s+1$.

Affine Gaudin model [Feigin, Frenkel'07]

- Consider r independent copies of the Kac-Moody $\widehat{\mathfrak{sl}}_{k_a}(2)$ algebra at levels $k_a = 1, 2, \ldots$.

$$J_A^{(a)}(u) J_B^{(b)}(0) = -\delta_{ab} \left(\frac{k_a}{2u^2} \, \eta_{AB} + \frac{\mathrm{i}}{u} \, f_{AB}{}^{C} \, J_C^{(a)} \right) + O(1) \, .$$

- To each copy one can associate the Virasoro field

$$G^{(a)} = \frac{\eta^{AB} J_A^{(a)} J_B^{(a)}}{k_a + 2} = \frac{1}{4\,(k_a+2)} \, \Big(J_0^{(a)} J_0^{(a)} + 2\, J_+^{(a)} J_-^{(a)} + 2\, J_-^{(a)} J_+^{(a)} \Big)$$

- Hamiltonians

$$\mathbf{H}_{\mathrm{G}}^{(a)} = \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \sum_{\substack{b=1 \\ b\neq a}}^{r} \frac{k_b \, G^{(a)} + k_a \, G^{(b)} - 2\eta^{AB} J_A^{(a)} J_B^{(b)}}{z_a - z_b}$$

- Conjecture [FF'07]: $\mathbf{H}_{\mathrm{G}}^{(a)} \in \{\mathbb{I}_s\} \longleftarrow$ infinite set of local IM

- **How to derive the Bethe Anzatc equations?**

Key idea for the “affinization” of the BA

A COURSE OF

MODERN ANALYSIS

AN INTRODUCTION TO THE GENERAL THEORY OF INFINITE PROCESSES AND OF ANALYTIC FUNCTIONS; WITH AN ACCOUNT OF THE PRINCIPAL TRANSCENDENTAL FUNCTIONS

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CHAPTER XXIII

ELLIPSOIDAL HARMONICS AND LAMÉ'S EQUATION

23·1. *The definition of ellipsoidal harmonics.*

It has been seen earlier in this work (§ 18·4) that solutions of Laplace's equation, which are analytic near the origin and which are appropriate for the discussion of physical problems connected with a sphere, may be conveniently expressed as linear combinations of functions of the type

$$r^n P_n(\cos\theta), \qquad r^n P_n^m(\cos\theta)\,{\cos \atop \sin}\, m\phi,$$

where n and m are positive integers (zero included).

When $P_n(\cos\theta)$ is resolved into a product of factors which are linear in $\cos^2\theta$ (multiplied by $\cos\theta$ when n is odd), we see that, if $\cos\theta$ is replaced by z/r, then the zonal harmonic $r^n P_n(\cos\theta)$ is expressible as a product of factors which are linear in x^2, y^2 and z^2, the whole being multiplied by z when n is odd. The tesseral harmonics are similarly resoluble into factors which are linear in x^2, y^2 and z^2 multiplied by one of the eight products $1, x, y, z, yz, zx, xy, xyz$.

The surfaces on which any given zonal or tesseral harmonic vanishes are surfaces on which either θ or ϕ has some constant value, so that they are circular cones or planes, the coordinate planes being included in certain cases.

When we deal with physical problems connected with ellipsoids, the structure of spheres, cones and planes associated with polar coordinates is replaced by a structure of confocal quadrics. The property of spherical harmonics which has just been explained suggests the construction of a set of harmonics which shall vanish on certain members of the confocal system.

Such harmonics are known as *ellipsoidal* harmonics; they were studied by Lamé* in the early part of the nineteenth century by means of confocal coordinates. The expressions for ellipsoidal harmonics in terms of Cartesian coordinates were obtained many years later by W. D. Niven†, and the following account of their construction is based on his researches.

The fundamental ellipsoid is taken to be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and any confocal quadric is

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1,$$

* *Journal de Math.* IV. (1839), pp. 100–125, 126–163.
† *Phil. Trans.* 182 A (1892), pp. 231–278.

where θ is a constant. It will be necessary to consider sets of such quadrics, and it conduces to brevity to write

$$\frac{x^2}{a^2+\theta_p}+\frac{y^2}{b^2+\theta_p}+\frac{z^2}{c^2+\theta_p}-1\equiv\Theta_p,\qquad \frac{x^2}{a^2+\theta_p}+\frac{y^2}{b^2+\theta_p}+\frac{z^2}{c^2+\theta_p}\equiv K_p.$$

The equation of any member of the set is then

$$\Theta_p=0.$$

The analysis is made more definite by taking the x-axis as the longest axis of the fundamental ellipsoid and the z-axis as the shortest, so that $a>b>c$.

23·2. *The four species of ellipsoidal harmonics.*

A consideration of the expressions for spherical harmonics in factors indicates that there are four possible species of ellipsoidal harmonics to be investigated. These are included in the scheme

$$\left\{\begin{matrix} & x, & yz, & \\ 1, & y, & zx, & xyz \\ & z, & xy, & \end{matrix}\right\}\Theta_1\Theta_2\ldots\Theta_m,$$

where one or other of the expressions in { } is to multiply the product $\Theta_1\Theta_2\ldots\Theta_m$.

If we write for brevity

$$\Theta_1\Theta_2\ldots\Theta_m=\Pi(\Theta),$$

any harmonic of the form $\Pi(\Theta)$ will be called *an ellipsoidal harmonic of the first species.* A harmonic of any of the three forms* $x\Pi(\Theta)$, $y\Pi(\Theta)$, $z\Pi(\Theta)$ will be called *an ellipsoidal harmonic of the second species.* A harmonic of any of the three forms* $yz\Pi(\Theta)$, $zx\Pi(\Theta)$, $xy\Pi(\Theta)$ will be called *an ellipsoidal harmonic of the third species.* And a harmonic of the form $xyz\Pi(\Theta)$ will be called *an ellipsoidal harmonic of the fourth species.*

The terms of highest degree in these species of harmonics are of degrees $2m$, $2m+1$, $2m+2$, $2m+3$ respectively. It will appear subsequently (§ 23·26) that $2n+1$ linearly independent harmonics of degree n can be constructed, and hence that the terms of degree n in these harmonics form a fundamental system (§ 18·3) of harmonics of degree n.

We now proceed to explain in detail how to construct harmonics of the first species and to give a general account of the construction of harmonics of the other three species. The reader should have no difficulty in filling up the *lacunae* in this account with the aid of the corresponding analysis given in the case of functions of the first species.

* The three forms will be distinguished by being described as different *types* of the species.

The linear independence of the $2n+1$ Lamé functions of degree n is therefore established.

23·45. *The linear independence of ellipsoidal harmonics.*

Let $G_n^m(x, y, z)$ be the ellipsoidal harmonic of degree n associated with $E_n^m(\xi)$, and let $H_n^m(x, y, z)$ be the corresponding homogeneous harmonic.

It is now easy to shew that not only are the $2n+1$ harmonics of the type $G_n^m(x, y, z)$ linearly independent, but also the $2n+1$ harmonics of the type $H_n^m(x, y, z)$ are linearly independent.

In the first place, if a linear relation existed between harmonics of the type $G_n^m(x, y, z)$. then, when we expressed these harmonics in terms of confocal coordinates (λ, μ, ν), we should obtain a linear relation between Lamé functions of the type $E_n^m(\xi)$ where $\xi = \lambda + \frac{1}{3}(a^2+b^2+c^2)$, and it has been seen that no such relation exists.

Again, if a linear relation existed between homogeneous harmonics of the type $H_n^m(x, y, z)$, by operating on the relation with Niven's operator (§ 23·25),

$$1 - \frac{D^2}{2(2n-1)} + \frac{D^4}{2\,.\,4\,(2n-1)(2n-3)} - \dots,$$

we should obtain a linear relation connecting functions of the type $G_n^m(x, y, z)$, and since it has just been seen that no such relation exists, it follows that the homogeneous harmonics of degree n are linearly independent.

23·46. *Stieltjes' theorem on the zeros of Lamé functions.*

It has been seen that any Lamé function of degree n is expressible in the form

$$(\theta+a^2)^{\kappa_1}(\theta+b^2)^{\kappa_2}(\theta+c^2)^{\kappa_3}\,.\prod_{p=1}^{m}(\theta-\theta_p),$$

where $\kappa_1, \kappa_2, \kappa_3$ are equal to 0 or $\frac{1}{2}$ and the numbers $\theta_1, \theta_2, \dots \theta_m$ are real and unequal both to each other and to $-a^2, -b^2, -c^2$; and $\frac{1}{2}n = m + \kappa_1 + \kappa_2 + \kappa_3$. When $\kappa_1, \kappa_2, \kappa_3$ are given the number of Lamé functions of this degree and type is $m+1$.

and consider the product

$$\Pi = \prod_{p=1}^{m}\left[|(\phi_p+a^2)|^{\kappa_1+\frac{1}{4}}\,.\,|(\phi_p+b^2)|^{\kappa_2+\frac{1}{4}}\,.\,|(\phi_p+c^2)|^{\kappa_3+\frac{1}{4}}\right]\prod_{p\neq q}|(\phi_p-\phi_q)|.$$

This product is zero when all the variables ϕ_p have their least values and also when all have their greatest values; when the variables ϕ_p are unequal both to each other and to $-a^2, -b^2, -c^2$, then Π is positive and it is obviously a continuous bounded function of the variables.

Hence there is a set of values of the variables for which Π attains its upper bound, which is positive and not zero (cf. § 3·62).

For this set of values of the variables the conditions for a maximum give

$$\frac{\partial \log \Pi}{\partial \phi_1} = \frac{\partial \log \Pi}{\partial \phi_2} = \dots = 0,$$

that is to say

$$\frac{\kappa_1+\frac{1}{4}}{\phi_p+a^2} + \frac{\kappa_2+\frac{1}{4}}{\phi_p+b^2} + \frac{\kappa_3+\frac{1}{4}}{\phi_p+c^2} + \sum_{q=1}^{m}{}' \frac{1}{\phi_p-\phi_q} = 0,$$

where p assumes in turn the values $1, 2, \dots m$.

Now this system of equations is precisely the system by which $\theta_1, \theta_2, \dots \theta_p$ are determined (cf. §§ 23·21–23·24); and *so the system of equations determining $\theta_1, \theta_2, \dots \theta_m$ has a solution for which*

$$\begin{cases} -a^2 < \theta_p < -b^2, & (p = 1, 2, \dots r-1) \\ -b^2 < \theta_p < -c^2. & (p = r, r+1, \dots m) \end{cases}$$

Hence, if r has any of the values $1, 2, \dots m+1$, a Lamé function exists with $r-1$ of its zeros between $-a^2$ and $-b^2$ and the remaining $m-r+1$ zeros between $-b^2$ and $-c^2$.

Since there are $m+1$ Lamé functions of the specified type, they are all obtained when r is given in turn the values $1, 2, \dots m+1$; and this is the theorem due to Stieltjes.

* *Acta Mathematica*, vi. (1885), pp. 321–326.

† The zeros $-a^2, -b^2, -c^2$ are to be omitted from this enumeration, $\theta_1, \theta_2, \dots \theta_m$ only being taken into account.

$$\sum_{a=1}^{r} \frac{\mathtt{j}_a}{z_a - x_m^{(+)}} - \sum_{\substack{n=1 \\ n\neq m}}^{\mathtt{M}_+} \frac{1}{x_n^{(+)} - x_m^{(+)}} = 0 \qquad (m = 1, 2, \dots, \mathtt{M}_+)$$

Relation to ODE [Gaudin'76, Sklyanin'87, E.Frenkel'04,...]

- $$\big(-\partial_z^2 + t_0(z)\big)\,\Psi = 0 \qquad \text{with} \qquad t_0(z) = \sum_{a=1}^{r}\left(\frac{\mathrm{j}_a(\mathrm{j}_a+1)}{(z-z_a)^2} + \frac{E_a}{z-z_a}\right)$$

  The ODE possesses r regular singular points at $z = z_a$. If

  $$\sum_{a=1}^{r} E_a = 0$$

  then $z = \infty$ is also a regular singularity so that the ODE is a Fuchsian one.

- If the residues $\{E_a\}_{a=1}^{r}$ in the potential $t_0(z)$ coincide with the set of energies corresponding to some common eigenvector of the Hamiltonians $\mathbf{H}^{(a)}$, then all the singular points at $z = z_a$ turn out to be apparent
  (A singularity z_a is called apparent if the ratio of any two solutions of the ODE is single valued in the vicinity of that point)

  $$\Psi_+(z) = \frac{\prod_{m=1}^{\mathtt{M}_+}(z - x_m^{(+)})}{\prod_{a=1}^{r}(z-z_a)^{\mathrm{j}_a}}$$

  is a solution of the ODE.

ODE/IQFT correspondence

There is a link between the spectrum of the Gaudin Hamiltonians and a class of differential equations possessing certain monodromy properties. This provides one of the simplest illustrations of a broad phenomena, known as the ODE/IQFT correspondence

[Voros'94; Dorey,Tateo'98; Bazhanov,Lukyanov,Zamolodchikov'98;...]

Affine Gaudin model [Feigin, Frenkel'07]

- $$J_A^{(a)}(u)J_B^{(b)}(0) = -\delta_{ab}\left(\frac{k_a}{2u^2}\ \eta_{AB} + \frac{\mathrm{i}}{u}\ f_{AB}{}^{C}\ J_C^{(a)}\right) + O(1)\ .$$

- $$G^{(a)} = \frac{\eta^{AB} J_A^{(a)} J_B^{(a)}}{k_a + 2} = \frac{1}{4\,(k_a+2)}\ \Big(J_0^{(a)} J_0^{(a)} + 2\, J_+^{(a)} J_-^{(a)} + 2\, J_-^{(a)} J_+^{(a)}\Big)$$

- Feigin and Frenkel put forward the conjecture, that the spectrum of

$$\mathbf{H}_{\mathrm{G}}^{(a)} = \frac{1}{2}\int_0^{2\pi}\frac{\mathrm{d}u}{2\pi}\ \sum_{\substack{b=1\\ b\neq a}}^{r}\frac{k_b\, G^{(a)} + k_a\, G^{(b)} - 2\eta^{AB} J_A^{(a)} J_B^{(b)}}{z_a - z_b}$$

would be encoded in a class of differential equations though they did not explain exactly how the spectrum would be extracted from the ODEs.

GAGM [Kotousov,Lukyanov'21]

- $$\mathbf{H}^{(a)}_{\rm gen} = \int_0^{2\pi} \frac{{\rm d}u}{2\pi} \left[\frac{\beta^2 K}{1-\beta^2}\, G^{(a)} + \frac{1}{4K}\, \frac{1-\beta}{1+\beta} \left(k_a \left(J_0^{(\rm tot)}\right)^2 - K\, J_0^{(a)} J_0^{(\rm tot)} \right) \right.$$
$$\left. - \sum_{\substack{b=1\\ b\neq a}}^{r} \frac{1}{z_a - z_b} \left(\frac{1}{4} (z_a + z_b)\, J_0^{(a)} J_0^{(b)} + z_a\, J_+^{(b)} J_-^{(a)} + z_b\, J_+^{(a)} J_-^{(b)} - k_a z_b\, G^{(b)} - k_b z_a\, G^{(a)} \right) \right]$$
  where
  $$J_0^{(\rm tot)} = \sum_{a=1}^{r} J_0^{(a)} \qquad \text{and} \qquad K = \sum_{a=1}^{r} k_a$$
- $\mathbf{H}^{(a)}_{\rm gen}$ depend on the parameter β. The Hamiltonians of the affine Gaudin model are obtained through a certain limiting procedure, which includes taking $\beta \to 1^-$.
- The ODE/IQFT correspondence for the model
- The spectrum of $\mathbf{H}^{(a)}_{\rm gen}$ for arbitrary $\beta \in (0,1)$

The ODE for GAGM

$$\big(-\partial_z^2 + t_{\rm L}(z) + \kappa^2\, \mathcal{P}(z) \big)\ \Psi = 0$$

where

$$t_{\rm L}(z) = -\frac{A^2+\frac{1}{4}}{z^2} + \sum_{a=1}^{r}\left(\frac{\mathrm{j}_a(\mathrm{j}_a+1)}{(z-z_a)^2} + \frac{z_a\gamma_a}{z(z-z_a)} \right) + \sum_{\alpha=1}^{\rm L}\left(\frac{2}{(z-w_\alpha)^2} + \frac{w_\alpha \Gamma_\alpha}{z(z-w_\alpha)} \right)$$

$$\mathcal{P}(z) = z^{-2+\xi\sum_{a=1}^{r} k_a}\ \prod_{a=1}^{r}(z-z_a)^{k_a} \qquad\qquad \left(\xi = \tfrac{\beta^2}{1-\beta^2}\right)$$

- BA equations = a set of conditions that all the singularities are apparent except for $z=0$ and $z=\infty$.
- Eigenvalues of the Hamiltonians:

$$E^{(a)} = \frac{2z_a\gamma_a}{k_a} - 2z_a \sum_{\beta=1}^{\rm L}\frac{1}{z_a-w_\beta} - \frac{2z_a}{k_a}\sum_{\substack{b=1\\ b\neq a}}^{r}\frac{d_a k_b + d_b k_a}{z_a - z_b} - \frac{2d_a}{k_a}\,\big(\xi K - 2\big) - 2d_0$$

Quantum transfer-matrices [BLZ'94]

- Universal R-matrix [Drinfeld'86]

$$\mathcal{R} \in U_q(\widehat{\mathfrak{b}}_+) \otimes U_q(\widehat{\mathfrak{b}}_-) \ : \quad \mathcal{R}^{12}\,\mathcal{R}^{13}\,\mathcal{R}^{23} = \mathcal{R}^{23}\,\mathcal{R}^{13}\,\mathcal{R}^{12}$$

$$\mathfrak{g} = \mathfrak{sl}(2) \ : \quad \{y_0, y_1, h_0, h_1\} \in U_q(\widehat{\mathfrak{b}}_+)\ , \qquad \{x_0, x_1, h_0, h_1\} \in U_q(\widehat{\mathfrak{b}}_-)$$

- Evaluation homomorphism $U_q(\widehat{\mathfrak{g}}) \mapsto U_q(\mathfrak{g})[\lambda, \lambda^{-1}]$:

$$y_0 \mapsto \lambda q^{\frac{\mathtt{h}}{2}}\, \mathtt{e}_+\ , \qquad y_1 \mapsto \lambda\, q^{-\frac{\mathtt{h}}{2}}\, \mathtt{e}_-\ , \qquad h_0 \mapsto \mathtt{h}\ , \qquad h_1 \mapsto -\mathtt{h}$$

$$\mathtt{h}, \mathtt{e}_\pm \in U_q\big(\mathfrak{sl}(2)\big) \ : \quad [\,\mathtt{h}, \mathtt{e}_\pm\,] = \pm 2\, \mathtt{e}_\pm\ , \qquad [\,\mathtt{e}_+, \mathtt{e}_-\,] = \frac{q^{\mathtt{h}} - q^{-\mathtt{h}}}{q - q^{-1}}$$

-

$$\boldsymbol{L}_\ell(\lambda) = \big(\pi_\ell(\lambda) \otimes 1\big)[\mathcal{R}] \qquad\qquad \big(\ell = \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots\big)$$

is a $U_q(\widehat{\mathfrak{b}}_-)$-valued $(2\ell+1)\times(2\ell+1)$ matrix whose entries depend on λ.

$$R_{\ell,\ell'}(\lambda_1/\lambda_2)\ \big(\boldsymbol{L}_\ell(\lambda_1)\otimes \mathbf{1}\big)\big(\mathbf{1}\otimes \boldsymbol{L}_{\ell'}(\lambda_2)\big) = \big(\mathbf{1}\otimes \boldsymbol{L}_{\ell'}(\lambda_2)\big)\big(\boldsymbol{L}_\ell(\lambda_1)\otimes \mathbf{1}\,\big)\, R_{\ell,\ell'}(\lambda_1/\lambda_2)$$

$$R_{\ell_1,\ell_2}(\lambda_1/\lambda_2) = \big(\pi_{\ell_1}(\lambda_1)\otimes \pi_{\ell_2}(\lambda_2)\big)[\mathcal{R}]$$

- $$R_{\ell,\ell'}(\lambda_1/\lambda_2)\ \big(\boldsymbol{L}_\ell(\lambda_1)\otimes\mathbf{1}\big)\big(\mathbf{1}\otimes\boldsymbol{L}_{\ell'}(\lambda_2)\big)=\big(\mathbf{1}\otimes\boldsymbol{L}_{\ell'}(\lambda_2)\big)\big(\boldsymbol{L}_\ell(\lambda_1)\otimes\mathbf{1}\big)\,R_{\ell,\ell'}(\lambda_1/\lambda_2)$$

  Using [Khoroshkin,Stolin,Tolstoi'94],

  $$\boldsymbol{L}_\ell(\lambda)=\big(\pi_\ell(\lambda)\otimes 1\big)[\mathcal{R}]=q^{\frac{1}{2}\mathtt{h}\,h_0}\,\Big[1+\lambda\,(q-q^{-1})\,(x_0\,q^{\frac{\mathtt{h}}{2}}\,\mathtt{e}_+ + x_1\,q^{-\frac{\mathtt{h}}{2}}\,\mathtt{e}_-)$$
  $$+\lambda^2\,\frac{q-q^{-1}}{q^2\,[2]_q}\Big((q^2-1)\,x_0^2\,(q^{\frac{\mathtt{h}}{2}}\,\mathtt{e}_+)^2+(q^2-1)\,x_1^2\,(q^{-\frac{\mathtt{h}}{2}}\,\mathtt{e}_-)^2$$
  $$+(q^2\,x_1x_0-x_0x_1)\,(q^{\frac{\mathtt{h}}{2}}\,\mathtt{e}_+)(q^{-\frac{\mathtt{h}}{2}}\,\mathtt{e}_-)+(q^2\,x_0x_1-x_1x_0)\,(q^{-\frac{\mathtt{h}}{2}}\,\mathtt{e}_-)(q^{\frac{\mathtt{h}}{2}}\,\mathtt{e}_+)\Big)$$

  $x_0, x_1\in U_q(\widehat{\mathfrak{b}}_-)$ obey the quantum Serre relations

  $$x_a^3x_b-[3]_q\,x_a^2x_bx_a+[3]_q\,x_ax_bx_a^2-x_bx_a^3=0\qquad (a,b=0,1)$$

  Also

  $$[h_0,x_0]=-[h_1,x_0]=-2x_0\,,[h_0,x_1]=-[h_1,x_1]=2x_1\,,[h_0,h_1]=0$$

- $$\boldsymbol{\tau}_\ell(\lambda)=\mathrm{Tr}_\ell\Big[q^{\frac{1}{2}\mathtt{h}\,h_0}\,\boldsymbol{L}_\ell(\lambda)\Big]\ :\quad \big[\boldsymbol{\tau}_\ell(\lambda),\,\boldsymbol{\tau}_{\ell'}(\lambda')\big]=0$$

Vertex operators representation of $U_q(\widehat{\mathfrak{b}}_-)$

An important case is when the generators x_0, x_1 are realized as the formal integrals over the vertex operators

$$x_0 = \int_0^{2\pi} \mathrm{d}u \, V_+(u) \, , \qquad x_1 = \int_0^{2\pi} \mathrm{d}u \, V_-(u)$$

- Braiding relation

$$V_{\sigma_1}(u_1) V_{\sigma_2}(u_2) = q^{2\sigma_1\sigma_2} \; V_{\sigma_2}(u_2) V_{\sigma_1}(u_1) \, , \quad u_1 > u_2$$

- Quasiperiodicity condition

$$V_\pm(u + 2\pi) = q^{-2} \; \Omega^{\pm 1} \; V_\pm(u)$$

$$\Omega \, V_\pm(u) \, \Omega^{-1} = q^{\pm 4} \; V_\pm(u) \, , \qquad \text{i.e.} \qquad \Omega = q^{2h_0}$$

Path-ordered exponential

$$\boldsymbol{L}_\ell(\lambda) = \big(\pi_\ell(\lambda)\otimes 1\big)[\mathcal{R}] = q^{\frac{1}{2}\mathtt{h}\, h_0}\,\Big[1 + \lambda\,(q - q^{-1})\,(x_0\, q^{\frac{\mathtt{h}}{2}}\, \mathrm{e}_+ + x_1\, q^{-\frac{\mathtt{h}}{2}}\, \mathrm{e}_-)$$

$$+\lambda^2\,\frac{q-q^{-1}}{q^2\,[2]_q}\bigg(\,(q^2-1)\,x_0^2\,\,(q^{\frac{\mathtt{h}}{2}}\,\mathrm{e}_+)^2 + (q^2-1)\,x_1^2\,\,(q^{-\frac{\mathtt{h}}{2}}\,\mathrm{e}_-)^2$$

$$+(q^2\,x_1x_0 - x_0x_1)\,\,(q^{\frac{\mathtt{h}}{2}}\,\mathrm{e}_+)(q^{-\frac{\mathtt{h}}{2}}\,\mathrm{e}_-) + (q^2\,x_0x_1 - x_1x_0)\,(q^{-\frac{\mathtt{h}}{2}}\,\mathrm{e}_-)(q^{\frac{\mathtt{h}}{2}}\,\mathrm{e}_+)\bigg) + \ldots\Big]$$

$$\Downarrow$$

$$\boldsymbol{L}_\ell(\lambda) = \Omega^{\frac{1}{4}\mathtt{h}}\sum_{m=0}^{\infty}\lambda^m \sum_{\sigma_1\ldots\sigma_m=\pm}\big(q^{\frac{\mathtt{h}}{2}\sigma_1}\mathrm{e}_{\,\sigma_1}\big)\ldots\big(q^{\frac{\mathtt{h}}{2}\sigma_m}\mathrm{e}_{\,\sigma_m}\big)\;J(\sigma_1,\ldots,\sigma_m)$$

$$J(\sigma_1,\ldots,\sigma_m) = \int\limits_{2\pi>u_1>u_2>\ldots>u_m>0} \mathrm{d}u_1\ldots\mathrm{d}u_m\; V_{\sigma_1}(u_1)\ldots V_{\sigma_m}(u_m)$$

$$\boxed{\boldsymbol{L}_\ell(\lambda) = \Omega^{\frac{1}{4}\mathtt{h}}\;\overleftarrow{\mathcal{P}}\exp\bigg(\lambda\int_0^{2\pi}\mathrm{d}u\,\Big(V_-(u)\;q^{+\frac{\mathtt{h}}{2}}\,\mathrm{e}_+ + V_+(u)\;q^{-\frac{\mathtt{h}}{2}}\,\mathrm{e}_-\Big)\bigg)}$$

Examples

- $V_\pm = \mathrm{e}^{\pm 2\mathrm{i}\beta\phi}$ [BLZ'94]

$$\phi(u_1)\phi(u_2) = -\tfrac{1}{2}\log(u_1-u_2) + O(1)$$

$$\phi(u+2\pi) = \phi(u) + 2\pi\hat{a}_0$$

Then

$$V_{\sigma_1}(u_1)V_{\sigma_2}(u_2) = q^{2\sigma_1\sigma_2}\ V_{\sigma_2}(u_2)V_{\sigma_1}(u_1)\ , \qquad u_1 > u_2$$

$$\Omega\, V_\pm(u)\,\Omega^{-1} = q^{\pm 4}\ V_\pm(u)\ , \qquad q = \mathrm{e}^{\mathrm{i}\pi\beta^2}\ , \qquad \Omega = \mathrm{e}^{4\pi\mathrm{i}\beta\hat{a}_0}$$

- $V_+ = \mathrm{e}^{+2\mathrm{i}\beta\phi}\ , \qquad V_- = \partial\theta\mathrm{e}^{-2\mathrm{i}\beta\phi}$

[Bazhanov,Kotousov,Koval,Lukyanov'20]

Parafermionic realization of $U_q(\widehat{\mathfrak{b}}_-)$ [Lukyanov'06]

$\widehat{\mathfrak{sl}}_k(2)$ algebra with $k = 1, 2, \ldots$:

$$J_+(u)J_-(0) = -\frac{k}{u^2} - \frac{\mathrm{i}}{u}\ J_0(0) + O(1)\ , \quad J_0(u)J_\pm(0) = \mp\frac{2\mathrm{i}}{u}\ J_\pm(0) + O(1)$$

$$J_0(u)J_0(0) = -\frac{2k}{u^2} + O(1)$$

It admits a realization in terms of the $\mathbb{Z}_k$ parafermionic fields $\psi_\pm$ and the chiral Bose field ϕ [Fateev,Zamolodchikov'86]

$$J_\pm = \sqrt{k}\ \psi_\pm\ \mathrm{e}^{\pm\frac{2\mathrm{i}\phi}{\sqrt{k}}}\ , \qquad J_0 = 2\sqrt{k}\ \partial\phi$$

$$\psi_\pm(u + 2\pi) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{k}}\ \big(\hat{\Omega}_k\big)^{\pm 1}\,\psi_\pm(u)$$

$\hat{\Omega}_k$ - the operator of the $\mathbb{Z}_k$ charge:

$$\hat{\Omega}_k\,\psi_\pm\,\big(\hat{\Omega}_k\big)^{-1} = \omega^{\pm 2}\,\psi_\pm\ , \qquad \omega = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{k}}$$

$$\boxed{V_\pm = \sqrt{k}\,\psi_\pm\ \mathrm{e}^{\pm\frac{2\mathrm{i}\beta\phi}{\sqrt{k}}}\ , \qquad q = -\mathrm{e}^{\frac{\mathrm{i}\pi}{k}(\beta^2-1)}\,, \qquad \Omega = \mathrm{e}^{\frac{4\pi\mathrm{i}\beta\hat{a}_0}{\sqrt{k}}}\ \hat{\Omega}_k}$$

W-algebra

- Let $\mathcal{L}^{(s)}$ be the linear space of chiral local fields of Lorentz spin s built out of the Bose field ϕ and the fundamental parafermions $\psi_\pm$.

- We choose one of the vertices, say V_+, and consider the linear subspace $\mathcal{W}^{(s)} \subset \mathcal{L}^{(s)}$ made up of local fields X_s such that

$$X_s(u)V_+(v) = \partial_v(\ldots) + O(1)$$

  In the physical slang, one says that the fields X_s commute with the "screening charge"

$$x_0 = \oint \mathrm{d}v \ V_+(v)$$

- Suppose X_s and $Y_{s'}$ both commute with x_0. By construction any local field which appears in the OPE $X_s(u)Y_{s'}(v)$ also commutes with the screening charge. Hence the direct sum $\oplus_{s\geq 1}\mathcal{W}^{(s)}$ possesses the structure of an operator algebra.

$WD(2,1|\alpha)$-algebra (corner brane W-algebra)

[Fateev'95; Feigin,Semikhatov'01; Lukyanov,Zamolodchikov'12]

- $\mathcal{L}^{(1)} = \mathrm{span}\big(\partial\phi\big)$, while $\mathcal{W}^{(1)} = \emptyset$
- $\mathcal{L}^{(2)} = \mathrm{span}\big((\partial\phi)^2,\, \partial^2\phi,\, W_2\big)$

$$\psi_+(u)\psi_-(v) = (u-v)^{-2+\frac{2}{k}}\left[1 + \tfrac{k+2}{2k}\ (u-v)^2\ \big(W_2(u)+W_2(v)\big) + \ldots\right]$$

$$\mathcal{W}^{(2)}\ \ :\quad X_2 = (\partial\phi)^2 + \tfrac{\mathrm{i}}{\sqrt{k}}\ \big(\beta^{-1}-\beta\big)\,\partial^2\phi + W_2$$

- $\mathcal{L}^{(3)} = \mathrm{span}\big((\partial\phi)^3,\, \partial^2\phi\partial\phi,\ \partial^3\phi,\, W_3,\, \partial\phi W_2,\, \partial W_2\big)$, $\mathcal{W}^{(3)} = \mathrm{span}\big(\partial X_2\big)$.
- $\mathcal{W}^{(4)} = \mathrm{span}\big(\partial^2 X_2, (X_2)^2, X_4\big)$

$$X_2(u)X_2(v) = \frac{c}{2(u-v)^2} - \frac{X_2(u)+X_2(v)}{(u-v)^2} + (X_2)^2(v) + O(u-v)$$

where $c = \frac{3k}{k+2} - \frac{6}{k}(\beta^{-1}-\beta)^2$.

- The construction can be simplified using bosonization for the parafermions

$$\psi_\pm = \Big(\partial\alpha \pm \mathrm{i}\sqrt{\tfrac{k+2}{k}}\ \partial\gamma\Big)\ \mathrm{e}^{\pm\frac{2\alpha}{\sqrt{k}}}\qquad \text{[Zamolodchikov'85, Wakimoto'86]}$$

X_4 is a certain differential polynomial built from $\partial\phi$, $\partial\alpha$, $\partial\gamma$.

Local IM

$T_{s+1} \in \mathcal{W}^{(s+1)}$:

$$T_{s+1}(u)V_-(v) = \sum_{m=2}^{s+1} \frac{R_{-m}(v)}{(u-v)^m} + \frac{R_{-1}(v)}{u-v} + O(1) \qquad \text{with} \qquad R_{-1} = \partial\mathcal{O}(v)$$

Then

$$\mathbb{I}_s = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\; T_{s+1}(u)$$

$$\left[\boldsymbol{\tau}_\ell(\lambda), \mathbb{I}_s\right] = \left[\,\mathbb{I}_s, \mathbb{I}_{s'}\right] = 0$$

In the case under consideration the operator $\mathbb{I}_s$ exists and is unique (up to overall normalization) only for odd $s = 1, 3, 5, \ldots$ [Fateev'95]

New realization of $U_q(\widehat{\mathfrak{b}}_-)$

- ϕ_a and $\psi^{(a)}_\pm$ are r independent copies of the chiral Bose and $\mathbb{Z}_{k_a}$ parafermionic fields

$$V^{(a)}_\pm = \sqrt{k_a}\ \psi^{(a)}_\pm\ \exp\Big[\pm 2\mathrm{i}\Big(\tfrac{\beta-1}{K}\sum_{\substack{b=1\\ b\neq a}}^r\sqrt{k_b}\ \phi_b + \big(\tfrac{\beta-1}{K}\ \sqrt{k_a}+\tfrac{1}{\sqrt{k_a}}\big)\ \phi_a\Big)\Big]$$

$K=\sum_{a=1}^r k_a$

-
$$V^{(a)}_\pm = J^{(a)}_\pm\ \mathrm{e}^{\pm\frac{2\mathrm{i}(\beta-1)}{\sqrt{K}}\varphi}$$

$$\varphi=\frac{1}{\sqrt{K}}\sum_{a=1}^r\sqrt{k_a}\,\phi_a\quad:\qquad J_0^{(\mathrm{tot})}\equiv\sum_{a=1}^r J^{(a)}_0 = 2\sqrt{K}\,\partial\varphi$$

-
$$x_0=\int_0^{2\pi}\mathrm{d}u\sum_{a=1}^r V^{(a)}_+\ ,\qquad x_1=\int_0^{2\pi}\mathrm{d}u\sum_{a=1}^r z_a\ V^{(a)}_-$$

$q=-\mathrm{e}^{\frac{\mathrm{i}\pi}{K}(\beta^2-1)}$

W-algebra

$$X_s(u)V_+(v) = \partial_v(\ldots) + O(1)$$

$$V_+ = \sum_{a=1}^{r} \sqrt{k_a}\ \psi_+^{(a)}\ \exp\left[2\mathrm{i}\left(\tfrac{\beta-1}{K}\sum_{\substack{b=1\\ b\neq a}}^{r}\sqrt{k_b}\ \phi_b + \left(\tfrac{\beta-1}{K}\ \sqrt{k_a} + \tfrac{1}{\sqrt{k_a}}\right)\phi_a\right)\right]$$

Conjecture I

There exists a non-trivial W-algebra

$$W_{\boldsymbol{k}}^{(c,r)}\ :\quad \boldsymbol{k} = (k_1, \ldots, k_r)\ ;\quad r = 1, 2, \ldots$$

c- the central charge of the Virasoro subalgebra generated by

$$T_2 = \sum_{a=1}^{r}\left((\partial\phi_a)^2 + \tfrac{\mathrm{i}\sqrt{k_a}}{K}\ \left(\beta^{-1} - \beta\right)\ \partial^2\phi_a + W_2^{(a)}\right)\ ,$$

$$c = \sum_{a=1}^{r}\frac{3k_a}{k_a+2} - \frac{6}{K}\ \left(\beta^{-1} - \beta\right)^2$$

Local IM

$$T_{s+1}(u)V_-(v) = \sum_{m=2}^{s+1} \frac{R_{-m}(v)}{(u-v)^m} + \frac{R_{-1}(v)}{u-v} + O(1) \qquad \text{with} \qquad R_{-1} = \partial\mathcal{O}(v)$$

$$V_- = \sum_{a=1}^{r} z_a \sqrt{k_a}\ \psi_-^{(a)}\ \exp\Big[-2\mathrm{i} \Big(\tfrac{\beta-1}{K} \sum_{\substack{b=1\\ b\neq a}}^{r} \sqrt{k_b}\ \phi_b + \big(\tfrac{\beta-1}{K}\ \sqrt{k_a} + \tfrac{1}{\sqrt{k_a}}\big)\ \phi_a \Big)\Big]$$

Conjecture II

$\forall n = 1, 2, 3, \ldots$ there exist r linear independent local IM

$$\mathbb{I}_{2n-1}^{(a)} = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi}\ T_{2n}^{(a)} \qquad a = 1, \ldots, r$$

$$\big[\boldsymbol{\tau}_\ell(\lambda), \mathbb{I}_{2n-1}^a\big] = \big[\mathbb{I}_{2n-1}^a, \mathbb{I}_{2m-1}^b\big] = 0$$

Hamiltonians of the GAGM $\mathbf{H}^{(a)}_{\rm gen} \equiv \mathbb{I}^{(a)}_1$

$$\mathbf{H}^{(a)}_{\rm gen} = \int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \Bigg[\frac{\beta^2 K}{1-\beta^2}\, G^{(a)} + \frac{1-\beta}{1+\beta}\, \Big(k_a\, (\partial\varphi)^2 - \sqrt{K k_a}\, \partial\phi_a \partial\varphi \Big)$$

$$+ \sum_{\substack{b=1\\ b\neq a}}^{r} \frac{1}{z_a - z_b} \Big(k_a z_b\, G^{(b)} + k_b z_a\, G^{(a)} \Big)$$

$$- \sum_{\substack{b=1\\ b\neq a}}^{r} \frac{\sqrt{k_a k_b}}{z_a - z_b} \Big((z_a + z_b)\, \partial\phi_a \partial\phi_b + z_a\, \psi_+^{(b)} \psi_-^{(a)} \mathrm{e}^{2\mathrm{i}(\phi_b - \phi_a)} + z_b\, \psi_+^{(a)} \psi_-^{(b)} \mathrm{e}^{2\mathrm{i}(\phi_a - \phi_b)} \Big) \Bigg]$$

where

$$G^{(a)} = (\partial\phi_a)^2 + W_2^{(a)}\,, \qquad\qquad \partial\varphi = \sum_{a=1}^{r} \sqrt{\frac{k_a}{K}}\, \partial\phi_a$$

Summary

- Generalzed affine Gaudin model

$$\mathbf{H}^{(a)}_{\rm gen}=\int_0^{2\pi}\frac{{\rm d}u}{2\pi}\left[\frac{\beta^2K}{1-\beta^2}\ G^{(a)}+\frac{1}{4K}\ \frac{1-\beta}{1+\beta}\ \Big(k_a\,\big(J_0^{({\rm tot})}\big)^2-K\,J_0^{(a)}J_0^{({\rm tot})}\Big)\right.$$

$$\left.-\sum_{\substack{b=1\\ b\neq a}}^{r}\frac{1}{z_a-z_b}\ \Big(\frac{1}{4}\,(z_a+z_b)\,J_0^{(a)}J_0^{(b)}+z_a\,J_+^{(b)}J_-^{(a)}+z_b\,J_+^{(a)}J_-^{(b)}-k_az_b\,G^{(b)}-k_bz_a\,G^{(a)}\Big)\right]$$

where

$$J_0^{({\rm tot})}=\sum_{a=1}^{r}J_0^{(a)}\qquad\qquad \text{and}\qquad\qquad K=\sum_{a=1}^{r}k_a$$

- The GAGM fits within the framework of the standard Yang-Baxter integrability. The Hamiltonians $\mathbf{H}^{(a)}_{\rm gen}$ are part of a large commuting family which, as usual, involves the quantum transfer-matrices and Baxter Q-operators.
- The ODE/IQFT correspondence
- The spectrum of $\mathbf{H}^{(a)}_{\rm gen}$ for arbitrary $\beta\in(0,1)$

Applications

- The GAGM governs the critical behaviour of a lattice system, which in the simplest case coincides with the inhomogeneous six-vertex model introduced by [Baxter'71]

  The local Boltzmann weights are contained in the R-matrix that is the trigonometric solution of the Yang-Baxter equation with the anisotropy parameter

  $$q = -\,\mathrm{e}^{\frac{\mathrm{i}\pi}{K}(\beta^2-1)}$$

  In the limit $\beta \to 1^-$ the trigonometric R-matrix becomes the rational one. ("Yangian limit")

- It is expected that GAGM integrable hierarchy includes the defformation of "integrable couples σ models" [Delduc,Lacroix,Margo,Vicedo'19]
  The results of this work provide an avenue for the systematic quantization of such theories within the ODE/IQFT correspondence.

- The GAGM may be understood as an integrable multiparametric generalization of the Kondo model [..., Gaiotto,Lee,Wu'20,...]