ODE/IM correspondence in the large momentum limit

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Based on works in collaboration with Davide Masoero

- R. Conti and D. Masoero, "*Counting monster potentials*", JHEP 2021
- R. Conti and D. Masoero, to appear.

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ODE in the complex plane

Schröedinger eq. $\psi''(x) = (V(x) - E)\,\psi(x)\ , \quad V(x) = x^{2\alpha}\ , \ \alpha \in \mathbb{R}$

Key idea (Sibuya & Voros, early 80s)

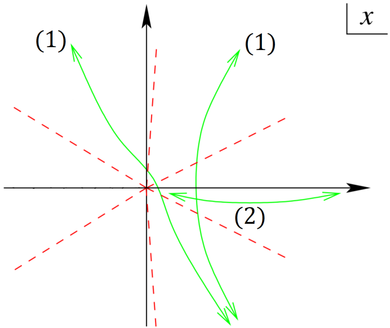
Examine the asymptotic behavior of a solution $\psi(x;E)$ as $|x| \to \infty$ along generic contours in the complex x plane, using WKB method

$\exists$ "basic" solution $\psi(x;E)$

- entire, asymptotics fixed by WKB
- (sub-)dominant in Stokes sectors

Spectral determinants

1. lateral connection problem (1)
   $\{C_n(E)\}$ (Stokes multipliers)
2. central connection problem (2)
   $D_{\pm}(E)$



Spectral determinants

Spectral determinants $\{C_n(E), D_\pm(E)\}$

1. are entire functions of E
2. have real zeroes (Dorey-Dunning-Tateo, '01)
3. have a well-known asymptotic behavior as $E \to \infty$
4. fulfill quadratic functional relations (Dorey-Tateo, '98)
   - T-system ($\{C_n\}$ only) $\overset{(2\alpha\in\mathbb{N})}{\Longleftrightarrow}$ **Thermodynamic Bethe Ansatz (TBA)**
   - quantum Wronskian ($D_\pm$ only) $\Longleftrightarrow$ **Bethe Ansatz equations (BAE)**

TBA & BAE

TBA: system of $2\alpha - 1$ coupled integral equations for $\{C_n(E)\}$
BAE: system of infinitely-many coupled algebraic equations for $\{E_k^\pm\}$

$$e^{\frac{\pi i}{\alpha+1}} \prod_{j=0}^{\infty} \frac{E_k^\pm - e^{\frac{\pi i}{\alpha+1}} E_j^\pm}{E_k^\pm - e^{-\frac{\pi i}{\alpha+1}} E_j^\pm} = -1 \,, \quad k \in \mathbb{N}$$

Integrable structure of CFTs and quantum KdV

Bazhanov-Lukyanov-Zamolodchikov approach (BLZ, '94,'96,'98)

Construct field theoretical versions of Baxter's $\mathbb{T}$ and $\mathbb{Q}$ (lattice) transfer matrices in Integrable QFTs

CFT with central charge $c < 1$ and highest weight (h.w.) Δ

$\mathcal{V}_\Delta$ = h.w. Virasoro module, $|\Delta\rangle$ = h.w. state (ground-state)

- $\exists$ commuting functions $\mathbf{T}_{n/2}(s), \mathbf{Q}_\pm(s) : \mathcal{V}_\Delta \to \mathcal{V}_\Delta$, $n \in \mathbb{N}$, $s \in \mathbb{C}$ (spectral parameter), continuum analogues of $\mathbb{T}$ and $\mathbb{Q}$
- $\{\mathbf{T}_{n/2}(s), \mathbf{Q}_\pm(s)\}$ generates KdV (non-)local Integrals of Motion (IM)

Fundamental observation (Dorey-Tateo, '98)

The ground-state eigenvalues $\{T^{[\Delta]}_{n/2}(s), Q^{[\Delta]}_\pm(s)\}$ have the same properties of the spectral determinants $\{C_n(E), D_\mp(E)\}$

ODE/IM correspondence

Add an angular momentum term $\frac{L}{x^2}$ with $L = l(l+1)$ (BLZ, '98)

Ground-state ODE/IM correspondence (Dorey-Tateo & BLZ, '98)

The ground-state eigenvalues $\{T^{[\Delta]}_{n/2}(s), Q^{[\Delta]}_{\pm}(s)\}$ are in bijection with the spectral determinants $\{C_n(E), D_{\mp}(E)\}$ of a Schröedinger eq. with potential

$$V_G(x) = x^{2\alpha} + \frac{L}{x^2}\,, \quad L \in \mathbb{C}$$

provided the following dictionary $(\alpha, L, E) \leftrightarrow (c, \Delta, s)$ holds

$$c = 1 - \frac{6\alpha^2}{\alpha+1}\,, \; \Delta = \frac{4L+1-4\alpha^2}{16(\alpha+1)}\,, \; s = f(\alpha)E \quad (*)$$

- solid for $2\alpha \in \mathbb{N}^*$
- expected to hold for any $\alpha \in \mathbb{R}^+$ (several numerical evidences)

Further developments

Conformal case

- higher-states of KdV (BLZ, '03 & Fioravanti, '04)
- CFTs with extended $\mathcal{W}$ symmetries (Dorey-Dunning-Tateo, '00 & Suzuki, '00)
- $\hat{\mathfrak{g}}-$KdV models: ODE$\to$ $\hat{\mathfrak{g}}^L$ affine opers (Feigin-Frenkel, '11 & Masoero-Raimondo-Valeri, '16,'17 & Masoero-Raimondo, '20)

Massive Integrable QFTs

- sine-Gordon (Lukyanov-Zamolodchikov, '10 & Fioravanti-Rossi, '21)
- affine Toda field theories (Adamopoulou-Dunning, '14)

Related problems

- potentials with arbitrary polynomials (Ito-Marino-Shu, '18)
- Hitchin systems (Gaiotto-Moore-Neitzke, '10 & Gaiotto, '14)

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Higher-states of quantum KdV

Let $N \in \mathbb{N}$ and $\nu = (\nu_1, \ldots, \nu_j)$ a partition of N with $\nu_{l+1} \leq \nu_l$

- level-N higher-states: $|\nu\rangle := L_{\nu_1} \ldots L_{\nu_j} |\Delta\rangle$ s.t. $L_0 |\nu\rangle = (\Delta + N) |\nu\rangle$
- $p(N)$ level-N higher-states, $p(N) =$ number of integer partitions of N

Conjecture (BLZ, '03)

Fix $N \in \mathbb{N}$. Then $\forall |\nu\rangle$, $\exists!$ monic polynomial $P(z) = \prod_{k=1}^{N}(z - z_k)$ s.t. the level-N eigenvalues $\{T^{[\nu]}_{n/2}(s), Q^{[\nu]}_{\pm}(s)\}$ are in bijection with the spectral determinants of a Schröedinger eq. with "monster" potential

$$V_P(x) = x^{2\alpha} + \frac{L}{x^2} - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log P(x^{2\alpha+2})$$

with trivial monodromy at $x = x_k := z_k^{\frac{1}{2\alpha+2}}$, provided $(*)$ holds

$(*) =$ Dorey-Tateo & BLZ dictionary $(\alpha, L, E) \leftrightarrow (c, \Delta, s)$

Trivial monodromy and BLZ system

- $\gamma =$ small loop in $\mathbb{C}$
- $\psi(x) =$ solution of the Schröedinger eq.
- $\psi_\gamma(x) =$ analytic continuation of $\psi(x)$ along γ

Trivial monodromy

$V_P(x)$ has trivial monodromy at $x = x_k$ if $\psi_\gamma(x) \equiv \psi(x)$ for any small loop γ around $x = x_k$

BLZ system

If $\{z_k\}_{k=1}^N$ are distinct, trivial monodromy condition $\iff$ **BLZ system**

$$\sum_{j\neq k} \frac{z_k^3 + (\alpha+3)(2\alpha+1)z_j z_k^2 + \alpha(2\alpha+1)z_j^2 z_k}{(z_k - z_j)^3} - \frac{\alpha z_k + \alpha^2 - L - \frac{1}{4}}{4(\alpha+1)} = 0$$

with $k = 1, \ldots, N$

The large L limit

Old problem

No direct relation between spectral determinants and eigenvalues of **T**- and **Q**-operators

Indirect relation through the BAE. However,

- BLZ system is very complicated in general (even at small N). Characterization of monster potentials is difficult
- BAE associated to monster potentials are even more complicated. Classification of BAE solutions is difficult

Feasible at large (positive) L

Strategy

Focus on the ODE side and study monster potentials at large L

Large L solutions of the BLZ system

BLZ system

$$\sum_{j\neq k} \frac{z_k^3 + (\alpha+3)(2\alpha+1)z_j z_k^2 + \alpha(2\alpha+1)z_j^2 z_k}{(z_k - z_j)^3} - \frac{\alpha z_k + \alpha^2 - L - \frac{1}{4}}{4(\alpha+1)} = 0$$

with $k = 1, \ldots, N$

First observation (C-Masoero, '20)

Solutions of the BLZ system are s.t. $z_k \underset{L\to\infty}{\sim} \frac{L}{\alpha}$. Therefore $x_k := z_k^{\frac{1}{2\alpha+2}}$ "condensate" about the minima of $V_G(x) = x^{2\alpha} + \frac{L}{x^2}$

- choose an arbitrary minimum m_0
- define a scaling variable $t = t(x) := a\,\varepsilon^{\frac{1-\alpha}{\alpha+1}}(x - m_0)$ with $\varepsilon = L^{-\frac{1}{4}}$
- define scaled roots $t_k = t\left(z_k^{\frac{1}{2\alpha+2}}\right)$

Monster potentials at large L

Second observation (C-Masoero, '20)

At small ε (large L) around a minimum m_0,

$$V_P(t;\varepsilon) = t^2 + \mathcal{O}(\varepsilon) - 2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log P\left(t + \mathcal{O}(\varepsilon)\right), \quad P(t) = \prod_{k=1}^{N}(t - t_k)$$

- $V_P(t;0)$ is a level-N **rational extension of the harmonic oscillator**

$$U_P(t) = t^2 - 2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log P(t)$$

- $V_P(t;\varepsilon)$ is a **perturbation** of a level-N rational extension

Simplification of the problem

At large L, study perturbation of rational extensions

Rational extensions of the harmonic oscillator

Rational extensions are fully characterized

Theorem (Oblomkov, '99)

U_P is a rational extension of level-N iff $\exists$ a partition $\nu = (\nu_1, \ldots, \nu_j)$ of N s.t. $U_P = U^{[\nu]}$

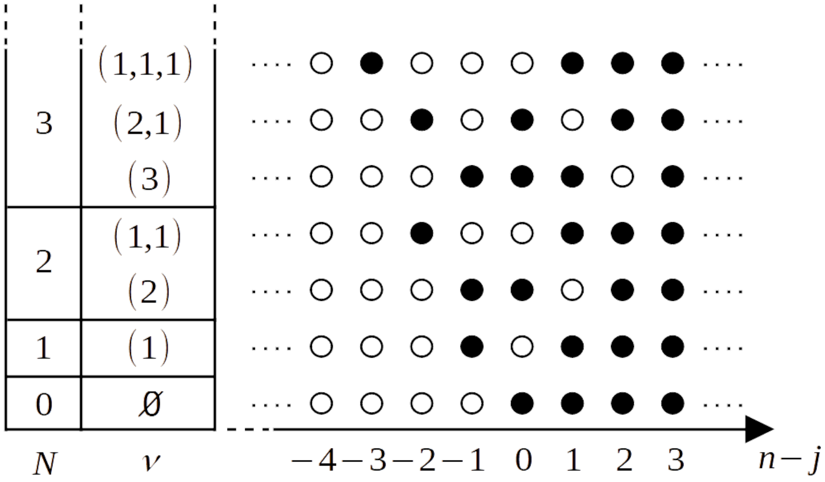
$$U^{[\nu]}(t) := t^2 - 2\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log P^{[\nu]}(t) \ , \ \ P^{[\nu]} := c_\nu \mathrm{Wr}\big[H_{\nu_j}, H_{\nu_{j-1}+1}, \ldots, H_{\nu_1+j-1}\big]$$

Moreover the spectrum of $\psi''(t) = (U^{[\nu]}(t) - \lambda)\,\psi(t)$ is

$$\lambda_n = 2(n-j)+1 \ , \quad n \in \mathbb{N}^{[\nu]} := \mathbb{N} \backslash \{\nu_j, \nu_{j-1}+1, \ldots, \nu_1+j-1\}$$

- for each N, $\exists\, p(N)$ rational extensions
- write $P^{[\nu]}(t) = \prod_{k=1}^{N} (t - v_k^{[\nu]})$

Pattern of energy levels



BLZ system at large L

At small ε (large L), BLZ system is

$$2t_k + \mathcal{O}(\varepsilon) - \sum_{j\neq k} \frac{4}{(t_k - t_j)^3} = 0\,, \quad k = 1, \dots, N$$

Problem

Show that, for ε small enough, the BLZ system admits a unique algebraic solution $\{t_k(\varepsilon)\}$ s.t. $\lim_{\varepsilon\to 0} t_k(\varepsilon) = v_k^{[\nu]}$, or equivalently with asymptotics

$$z_k^{[\nu]}(L) \underset{L\to\infty}{\sim} \frac{L}{\alpha} + \frac{(2\alpha+2)^{\frac{3}{4}} v_k^{[\nu]}}{\alpha} L^{-\frac{3}{4}}\,, \quad k = 1, \dots, N$$

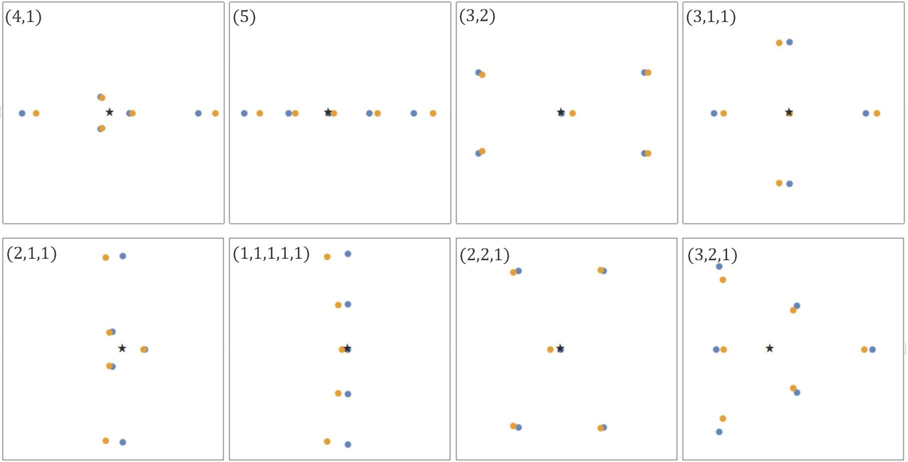
for any partition ν of N

- relatively easy for "non-degenerate" partitions ($\{v_k^{[\nu]}\}$ all distinct)
- highly non-trivial for "degenerate" partitions ($t = 0$ multiplicity > 1)

Solutions at large L

$\star = \frac{L}{\alpha} \qquad (L = 7 \times 10^4\ ,\ \alpha = \pi/3)$

● = exact solution (numerics) ● = aymptotic solution $z_k^{[\nu]}(L)$



Counting monster potentials

Theorem (C-Masoero, '20)

Fix $N \in \mathbb{N}$. For L large enough

- $\exists\, p(N)$ monster potentials with N roots
- for any partition $\nu = (\nu_1, \dots, \nu_j)$ of N, $!\exists$ monster potential s.t.

$$E_n^{[\nu]}(L) \underset{L \to +\infty}{\sim} (1+\alpha)\left(\frac{L}{\alpha}\right)^{\frac{\alpha}{\alpha+1}} + \left(2\alpha+2\right)^{\frac{1}{2}} \alpha^{\frac{1}{\alpha+1}} L^{\frac{\alpha-1}{2\alpha+2}} \left(2(n-j)+1\right)$$

with $n \in \mathbb{N}^{[\nu]} := \mathbb{N} \backslash \{\nu_j, \nu_{j-1}+1, \dots, \nu_1 + j - 1\}$

Summary & work in progress

We addressed the ODE/IM correspondence for KdV at large L and showed

- $\#$ N **roots monster potentials** $=$ $\#$ **level-**N **KdV higher-states**
- asymptotics of BAE solutions associated to monster potentials

To prove the full ODE/IM correspondence at large L we must construct the bijection

N **roots monster potentials** $\leftrightarrow$ **level-**N **KdV higher-states**

using the BAE as indirect link

Preliminary result (C-Masoero, to appear)

For L large enough, the BAE admits a unique solution with the same asymptotics $E_n^{[\nu]}$ (see previous slide), for any partition ν of N.

Outlook

The large momentum limit is a promising tool to study the mathematical structure underlying the ODE/IM correspondence.

It can also be applied to other models, e.g.

- $\hat{\mathfrak{g}}$-KdV models (exists the analogous of the BLZ system, but more complicated)
- massive models

of which little is known

Thank you for the attention